Geometric optics expansions for linear hyperbolic boundary value problems and optimality of energy estimates for surface waves.

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Antoine Benoit

Université de Nantes, Laboratoire de Mathématiques Jean Leray (CNRS UMR6629)
2 rue de la Houssinière, BP 92208, 44322 Nantes Cedex 3, France
Email: antoine.benoit@univ-nantes.fr

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Abstract

In this article we are interested in energy estimates for initial boundary value problem when surface waves occur that is to say when the uniform Kreiss Lopatinskii condition fails in the elliptic region or in the mixed region. More precisely we construct rigorous geometric optics expansions for elliptic and mixed frequencies and we show using those expansions that the instability phenomenon is higher in the case of mixed frequencies even if the uniform Kreiss Lopatinskii condition does not fail on hyperbolic modes. As a consequence this result allow us to give a classification of weakly well posed initial boundary value problems according to the region where the uniform Kreiss Lopatinskii condition degenerates.

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1 Introduction

In this article we will construct rigorous geometric optics expansions for solutions to a linear hyperbolic initial boundary value problem (ibvp in short) with constant coefficients.
The uniform Kreiss-Lopatinskii condition (UKL in short) is the main point in the study of initial boundary value problem. Indeed it is a necessary and sufficient condition for strong well-possedness of the ibvp. This condition due to Kreiss in [9] means that in the normal modes analysis no stable mode satisfies the homogeneous boundary conditions. In this article we are interested in ibvp for which this condition is not satisfied and fails for some "boundary" frequencies. The description of those frequencies will be made precise in section 2.2.

Geometric optics expansions for the ibvp are also highly linked to the structure of the resolvent matrix of the ibvp for boundary frequencies. Let us recall that thanks to the block structure of the resolvent equation (see theorem 2.1,[4], [3]) we know that there are four different kinds of frequencies, namely : elliptic, hyperbolic, mixed or glancing.
There are already many articles about geometric optics expansions for ibvp according to the frequency of oscillations in the source term, the fact that the ibvp is characteristic or not, the stability assumption on the boundary ... For instance, on the one hand, under the uniform Kreiss-Lopatinskii condition (UKL), Williams treats in [17] the case of a noncharacteristic ibvp for all possible frequencies, and Lescarret [11] deals with characteristic problems for a mixed-frequency. On the other hand, Marcou [12] and Coulombel-Guès [7] build expansions without UKL but for an elliptic frequency and a hyperbolic frequency respectively.
In this paper we are interested in noncharacteristic ibvp, when UKL fails in the elliptic region or in the mixed region of the boundary of the frequencies set (see section 2.1 for more details). So this paper will generalize Marcou's work to non homogenous problems and generalize Lescarret's work to problems for which UKL is violated. To some extent this paper will conclude the construction of geometric optics expansion for weakly well-posed ibvp.
The main consequence of the geometric optics expansions will be to describe the influence of the region in which UKL fails on the energy estimate that we have on the solution of the ibvp.
Indeed, in [9] Kreiss shows that UKL is a necessary and sufficient condition for the strong well-possedness of the corresponding ibvp in L^2. The corresponding energy estimate associated to the solution u is of the form :

$$\gamma\|u\|^2_{L^2_\gamma} + \|u_{|x_d=0}\|^2_{L^2_\gamma} \leq C\left(\frac{1}{\gamma}\|f\|^2_{L^2_\gamma} + \|g\|^2_{L^2_\gamma}\right),$$

where f is the source term in the interior and g the source term on the boundary.
Then, Sablé-Tougeron showed in [16] that if UKL fails at the order one in the elliptic region of the boundary of the space of frequencies, it is possible to consider solutions in a weaker sense. That is to say that there exists a unique solution of the ibvp that satisfies an energy estimate with a loss of one derivative on the boundary.
At last Coulombel treated in a more recent work cases where UKL is violated in the mixed region and in the hyperbolic region and showed two different energy estimates ([6] [5]). In the mixed region he proved that there is a loss of one derivative on the boundary (as in Sablé-Tougeron's work) and also a loss of half a derivative in the interior. Whereas when UKL fails in the hyperbolic region, there is a loss of one derivative both on the boundary and in the interior.
The question is to know whether those energy estimates are sharp.
In 2010, using geometric optics expansion for an ibvp with a failure of UKL in the hyperbolic region Coulombel-Guès [7] proved that the corresponding energy estimate is sharp.
This paper will treat two of the three remaining behaviours for the failure of

UKL, that is to say elliptic and mixed case. And we will be able to show the optimality of the energy estimates found by Sablé-Tougeron and Coulombel. In particular, our work shows that away from glancing points, the only possible failure of the UKL that allows for a homogeneous ibvp to be well-posed is in the elliptic region. This makes rigorous the discussion in ([3] , chapter 7) see also [15].

2 Notations, assumptions and main results.

2.1 Notations.

In this article we will consider ibvps in the half-space

$$\mathbb{R}^d_+ := \left\{x = (x', x_d) \in \mathbb{R}^d \setminus x_d \geq 0\right\}$$

and will we also note for $T > 0$, $\Omega_T :=]-\infty; T] \times \mathbb{R}^d_+$ and finally the spatial boundary of Ω_T will be denoted by ω_T.
Our typical ibvp will read :

$$\begin{cases} L(\partial)u^\varepsilon := \partial_t u^\varepsilon + \sum_{j=1}^d A_j \partial_j u^\varepsilon = f^\varepsilon \text{ in } \Omega_T, \\ Bu^\varepsilon_{|x_d=0} = g^\varepsilon \text{ in } \omega_T, \\ u^\varepsilon_{|t\leq 0} = 0, \end{cases} \tag{1}$$

where the A_j are square matrices of dimension N with real coefficients and B is a real matrix of dimension $p \times N$. The value of p will be made precise in assumption 2.3.
The superscript ε is sometimes used in order to highlight the main frequency of oscillations of the source terms and the solution in (1).
Energy estimates require the introduction of weighted Sobolev spaces. For s real and $\gamma > 0$ we will denote by $H^s_\gamma(\Omega_T)$ the weighted Sobolev space defined as follows :

$$H^s_\gamma(\Omega_T) = \left\{u \in \mathcal{D}'(\Omega_T) \setminus e^{-\gamma t} u \in H^s(\Omega_T)\right\},$$

and we also define $H^s_\gamma(\omega_T)$ and $L^2_{x_d}(H^s_\gamma(\omega_T))$ in the same spirit.
We introduce a partition of the frequencies space :

$$\begin{aligned} \Xi &= \left\{\zeta = (\sigma = \gamma + i\tau, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1}, \gamma \geq 0\right\}, \\ \Xi_0 &= \left\{\zeta \in \Xi, \gamma = 0\right\}. \end{aligned}$$

Let $\mathcal{A}(\zeta)$ be the resolvent matrix obtained after Fourier-Laplace transformation in the evolution equation of (1) :

$$\mathcal{A}(\zeta) = -A_d^{-1} \left(\sigma I + i \sum_{j=1}^{d-1} \eta_j A_j\right). \tag{2}$$

We denote by $E_-(\zeta)$ the stable subspace of $\mathcal{A}(\zeta)$, and $E_+(\zeta)$ the unstable subspace.

Thanks to Hersh's lemma (see [3] page 125) we know that for all σ such that $\mathcal{R}e\ \sigma > 0$, $\mathcal{A}(\zeta)$ does not have any purely imaginary eigenvalue and that $dim E_-(\zeta) = p$.
However if $\mathcal{R}e\sigma$ is zero, Hersh's lemma is not true anymore and has to be substituted by the following result which is due to Kreiss [9] and adapted by Métivier [13] for constantly hyperbolic operators that is to say the following assumption is satisfied.

Assumption 2.1 *There exist an integer $q \geq 1$, real valued analytic functions $\lambda_1, ..., \lambda_q$ on $\mathbb{R}^d \setminus \{0\}$and positive integers $\nu_1, ..., \nu_q$ such that :*

$$\forall \xi \in \mathbb{S}^{d-1}, det \left(\tau + \sum_{j=1}^{d} \xi_j A_j \right) = \prod_{k=1}^{q} (\tau + \lambda_k(\xi))^{\nu_k},$$

with $\lambda_1(\xi) < ... < \lambda_q(\xi)$ and the eigenvalues $\lambda_k(\xi)$ of $\sum_{k=1}^{d} A_k$ are semi-simple.

Theorem 2.1 *[Block structure] If the ibvp (1) satisfies assumption 2.1 then for all $\underline{\zeta}$ in Ξ there is a neighborhood $\mathcal{V}$ of $\underline{\zeta}$ in Ξ, an integer $L \geq 1$, a partition $N = \nu_1 + ... + \nu_L$ such that all $\nu_i \geq 1$, and a regular invertible matrix T defined on $\mathcal{V}$ such that we have :*

$$\forall \zeta \in \mathcal{V},\ T(\zeta)^{-1} \mathcal{A}(\zeta) T(\zeta) = diag \left(\mathcal{A}_1(\zeta), ..., \mathcal{A}_L(\zeta) \right),$$

where the size of $\mathcal{A}_i(\zeta)$ is ν_i, and $\mathcal{A}_i(\zeta)$ satisfies one of the following properties :
$i)$ All elements in the spectrum of $\mathcal{A}_i(\underline{\zeta})$ have a strictly negative real part .
$ii)$ All elements in the spectrum of $\mathcal{A}_i(\underline{\zeta})$ have a strictly positive real part .
$iii)$ $\nu_i = 1$, $\mathcal{A}_i(\underline{\zeta}) \in i\mathbb{R}$, $\partial_\gamma \mathcal{A}_i(\underline{\zeta}) \in \mathbb{R} \setminus \{0\}$
$iv)$ $\nu_i > 1$, $\exists k_i \in i\mathbb{R}$ such that

$$\mathcal{A}_i(\underline{\zeta}) = \begin{bmatrix} k_i & i & 0 \\ & \ddots & i \\ 0 & & k_i \end{bmatrix},$$

and the coefficient in the down left corner of $\partial_\gamma \mathcal{A}_i(\underline{\zeta}) \in \mathbb{R} \setminus \{0\}$.

Thanks to this theorem we are able to describe the four kinds of frequencies $\underline{\zeta}$ in Ξ_0.

Definition 2.1 *Let :*
• *$\mathcal{E}$ be the set of elliptic frequencies, that is to say the set of $\underline{\zeta}$ in Ξ_0 such that theorem 2.1 is satisfied with one block of type $i)$ and consequently also one of type $ii)$ only.*
• *$\mathcal{H}$ be the set of hyperbolic frequencies, that is to say the set of $\underline{\zeta}$ in Ξ_0 such that theorem 2.1 is satisfied with blocks of type $iii)$ only.*
• *$\mathcal{EH}$ be the set of mixed frequencies, that is to say the set of $\underline{\zeta}$ in Ξ_0 such that*

theorem 2.1 is satisfied with one block of each type i) and consequently also one of type ii) and at least one of type iii), but no block of type iv)
• *$\mathcal{G}$ be set the of glancing frequencies, that is to say the set of $\underline{\zeta}$ in Ξ_0 such that theorem 2.1 is satisfied with at least one block of type iv).*

We have the following partition of Ξ_0 :

$$\Xi_0 = \mathcal{E} \cup \mathcal{EH} \cup \mathcal{H} \cup \mathcal{G}.$$

The analysis in [[9],[13]] shows that $E_\pm(\zeta)$ admit a continous extension for the frequencies in Ξ_0. Moreover if, $\underline{\zeta} \in \Xi_0 \setminus \mathcal{G}$ then we can write

$$\mathbb{C}^N = E_-(\underline{\zeta}) \oplus E_+(\underline{\zeta}),$$

and

$$E_\pm(\underline{\zeta}) = E^e_\pm(\underline{\zeta}) \oplus E^h_\pm(\underline{\zeta}), \tag{3}$$

where $E^e_-(\underline{\zeta})$ $(resp.\ E^e_+(\underline{\zeta}))$ is the generalized space associated to the eigenvalues of $\mathcal{A}(\underline{\zeta})$ of negative $(resp.$ positive) real part, and $E^h_\pm(\underline{\zeta})$ are sums of eigenspaces of $\mathcal{A}(\underline{\zeta})$ associated to some purely imaginary eigenvalues of $\mathcal{A}(\underline{\zeta})$.
In fact, it is possible to give a precise decomposition of $E^h_\pm(\underline{\zeta})$. Let $i\underline{\omega}_m$ be a purely imaginary eigenvalue of $\mathcal{A}(\underline{\zeta})$, then

$$det(\underline{\tau} I + A(\underline{\eta}, \underline{\omega}_m)) = 0.$$

Using the constant hyperbolicity of $L(\partial)$ (see assumption 2.1), there is an index k_m such that

$$\underline{\tau} + \lambda_{k_m}(\underline{\eta}, \underline{\omega}_m) = 0, \tag{4}$$

with $\lambda_{k_m}(\eta, \omega_m)$ smooth.

Definition 2.2 *The set of causal (resp. noncausal) indeces $\mathcal{C}$ (resp. $\mathcal{NC}$) is defined by : m is in $\mathcal{C}$ (resp. $\mathcal{NC}$) if and only if for λ_{k_m} defined in (4) we have $\partial_{\eta_d}\lambda_{k_m}(\underline{\eta}, \underline{\omega}_m) > 0$ (resp. $\partial_{\eta_d}\lambda_{k_m}(\underline{\eta}, \underline{\omega}_m) < 0$).*
$v_m := \nabla\lambda_{k_m}(\underline{\eta}, \underline{\omega}_m)$ is called the group velocity associated to the phase $\underline{\tau} + \underline{\eta}.x' + \omega_m x_d$.

With such notations, we have

Lemma 2.1 *For all $\underline{\zeta} \in (\mathcal{EH} \cup \mathcal{H})$*

$$\begin{aligned} E^h_-(\underline{\zeta}) &= \oplus_{m\in\mathcal{C}} ker\mathcal{L}(\underline{\tau}, \underline{\eta}, \underline{\omega}_m), \\ E^h_+(\underline{\zeta}) &= \oplus_{m\in\mathcal{NC}} ker\mathcal{L}(\underline{\tau}, \underline{\eta}, \underline{\omega}_m), \end{aligned}$$

where for all ξ in $\mathbb{R}^{d+1}$, $\mathcal{L}(\xi_0, ..., \xi_d) = \xi_0 I + \sum_{j=1}^d \xi_j A_j$.

We refer to [7] for the proof.

Definition 2.3 *Set $\Pi_e^{\pm} := \Pi_e^{\pm}(\zeta)$ the projector on $E_{\pm}^e(\zeta)$ associated to the decomposition (3).*
Set $\Pi_m := \Pi_m(\zeta)$ the projector on $ker\mathcal{L}(\zeta, \omega_m)$ and $Q_m := Q_m(\zeta)$ the partial inverse of $\mathcal{L}(\zeta, \omega_m)$ satisfying :

$$\left\{ \begin{array}{l} Q_m\mathcal{L}(\zeta, \omega_m) = I - \Pi_m, \\ \Pi_m Q_m = Q_m \Pi_m = 0. \end{array} \right.$$

2.2 Assumptions.

Let us deal with non characteristic ibvps with constant multiplicity that is assumption 2.1 and the following assumption are checked.

Assumption 2.2 *The matrix A_d is invertible.*

The assumptions on the boundary condition are summarized in the following one :

Assumption 2.3 *B is a matrix of maximal rank p, with $p \geq 1$ the number of positive eigenvalues of A_d counted with multiplicity.*

Our last assumption explains how the uniform Kreiss-Lopatinskii condition degenerates. There are two distincts cases depending on the region of degeneration :

Assumption 2.4 • *The Kreiss-Lopatinskii condition is satisfied, that is to say for all $\zeta \in \Xi \setminus \Xi_0$, $kerB \cap E_-(\zeta) = \{0\}$.*
• *Let $\underline{\zeta} \in \Xi_0$ such that $kerB \cap E_-(\underline{\zeta}) \neq \{0\}$ then $\underline{\zeta} \in \mathcal{E}$ and there exist such frequencies.*
• *Moreover let $\underline{\zeta} \in \mathcal{E}$ such that $kerB \cap E_-(\underline{\zeta}) \neq \{0\}$, then there is a neighborhood $\mathcal{V}$ of $\underline{\zeta}$ in Ξ, a regular basis $(E_1, ..., E_p)(\zeta)$ of $E_-(\zeta)$, a regular and invertible matrix $P(\zeta)$ of size p and at last a regular real valued function θ such that we can write :*

$$\forall \zeta \in \mathcal{V},\ B\left[E_1, ..., E_p\right](\zeta) = P(\zeta) diag(\gamma + i\theta(\zeta), 1, ..., 1).$$

Assumption 2.5 • *The Kreiss-Lopatinskii condition is satisfied.*
• *Let $\underline{\zeta} \in \Xi_0$ such that $kerB \cap E_-(\underline{\zeta}) \neq \{0\}$ then $\underline{\zeta} \in \mathcal{EH}$ and such frequencies exist.*
• *Let $\underline{\zeta} \in \mathcal{EH}$ such that $kerB \cap E_-(\underline{\zeta}) \neq \{0\}$ then :*
⋄ *$E_-(\underline{\zeta}) \cap kerB = E_-^e(\underline{\zeta}) \cap kerB$.*
⋄ *There is a neighborhood $\mathcal{V}$ of $\underline{\zeta}$ in Ξ, a regular basis $(E_1^e, ..., E_{p-r}^e)(\zeta)$ of $E_-^e(\zeta)$,an inversible and regular matrix $P(\zeta)$ of size p and finally a regular real valued function θ such that :*

$$\forall \zeta \in \mathcal{V},\ B\left[E_1^e, ..., E_{p-r}^e, E_1^h, ..., E_r^h\right](\zeta) = P(\zeta) diag(\gamma + i\theta(\zeta), 1, ..., 1),$$

where $(E_1^h, ..., E_r^h)(\zeta)$ is a regular basis of $E_-^h(\zeta)$.

Assumptions 2.4 and 2.5 correspond to situations where the uniform Kreiss-Lopatinskii condition breaks down because of surface waves that have exponential decay with respect to the normal variable x_d. It is equivalent to assume that a Lopatinskii determinant vanishes at the order one for elliptic or mixed frequencies (see [2], [16]). However, assumption 2.5 considers a situation where hyperbolic modes also occur, even though they are not responsible for the breakdown of the uniform Kreiss-Lopatinskii condition.
One example of an ibvp which satisfies the assumption 2.4 is the onset of Rayleigh waves for the equations of elastodynamics. That is to say that when UKL fails in the elliptic area one can see waves which are localized along the boundary.
One example of physical interest we know of an ibvp which satisfies the assumption 2.5 is the liquid-vapor phase transition model especially studied by Benzoni-Gavage in [1] and later on by Coulombel in [6]. We provide in section 5 with another such example arising from the linearization of the Euler equations on a fixed domain.

Until the end we will work under one and only one of the two previous assumptions.

Now thanks to one of these assumptions we can describe the form of the source terms we are interested in.

Definition 2.4 • *Let P_{os} be the set of oscillating profiles ie functions $u(t,x,X_d)$ in $\mathcal{C}^\infty(\Omega_T \times \mathbb{R}_+)$ which can be written :*

$$u(t,x,X_d) = \sum_{m=1}^{M} e^{i\omega_m X_d} u_m(t,x),$$

with $(\omega_1, ..., \omega_M) \in \mathbb{R}^M$, $(u_1, ..., u_M) \in (H^{+\infty}(\Omega_T))^M$.
• *The set of functions $U(t,x,X_d)$ in $H^{+\infty}(\Omega_T \times \mathbb{R}_+)$ for which there is a positive δ such that $e^{\delta X_d} U(t,x,X_d)$ is in $H^{+\infty}(\Omega_T \times \mathbb{R}_+)$, will be denoted by P_{ev} and will be the set of evanescent profiles.*
• *To conclude, the set of profiles P is defined as follows $P = P_{os} \oplus P_{ev}$.*

Source terms used in the geometric optics expansion will not be the same under assumption 2.4 or under assumption 2.5. We shall be more specific later on.
To conclude this section we define the useful following vectors :

Definition 2.5 *Under assumption 2.4 or 2.5 there exists :*
• *a vector $e \in \mathbb{C}^N \setminus \{0\}$ such that $E_-(\underline{\zeta}) \cap kerB = E_-^e(\underline{\zeta}) \cap kerB = vect(e)$.*
• *A vector $b \in \mathbb{C}^p \setminus \{0\}$ such that $b.Bw = 0$, for all $w \in E_-(\underline{\zeta})$.*
• *$E_-(\underline{\zeta}) = vect(e) \oplus \breve{E}_-(\underline{\zeta})$ and B is an isomorphism from $\breve{E}_-(\underline{\zeta})$ to $b^\perp$.*

2.3 Main results.

As mentioned in the introduction energy estimates depend on the region in which the uniform Kreiss-Lopatinskii condition fails. In the case of a frequency

$\underline{\zeta}$ that satisfies the assumption 2.5 we will need the extra assumption :

Assumption 2.6 *If $\underline{\zeta}$ satisfies 2.5, then $E_+^h(\underline{\zeta})$ is non trivial.*

Let us recall the different energy estimates :

Theorem 2.2 *[16] [Sablé-Tougeron] Under assumptions 2.1-2.2-2.3-**2.4**, for all f in $L^2(\Omega_T)$, $g \in H^{\frac{1}{2}}(\omega_T)$ that vanish for $t \leq 0$, then there is a unique solution $u \in L^2(\Omega_T)$ of the ibvp (1) which moreover admits the following energy estimate :*

$$\|u\|^2_{L^2_\gamma(\Omega_T)} + \|u_{|x_d=0}\|^2_{H_\gamma^{-\frac{1}{2}}(\omega_T)} \leq \frac{C}{\gamma^2}\left(\|f\|^2_{L^2_\gamma(\Omega_T)} + \|g\|^2_{H_\gamma^{\frac{1}{2}}(\omega_T)}\right), \tag{5}$$

with γ sufficiently large and where C is independent of γ.
*[6] [Coulombel] Under assumptions 2.1-2.2-2.3-**2.5**-2.6, for all f in $L^2_{x_d}(H^{\frac{1}{2}}(\omega_T))$, $g \in H^{\frac{1}{2}}(\omega_T)$ that vanish for $t \leq 0$, then there is a unique solution $u \in L^2(\Omega_T)$ of the ibvp (1) which moreover admits the following energy estimate :*

$$\|u\|^2_{L^2_\gamma(\Omega_T)} + \|u_{|x_d=0}\|^2_{H_\gamma^{-\frac{1}{2}}(\omega_T)} \leq \frac{C}{\gamma^2}\left(\frac{1}{\gamma}\|f\|^2_{L^2_{x_d}(H_\gamma^{\frac{1}{2}}(\omega_T))} + \|g\|^2_{H_\gamma^{\frac{1}{2}}(\omega_T)}\right), \tag{6}$$

with γ sufficiently large and where C is independent of γ.

The estimates (5)-(6) are true for γ above a threshold γ_0, and it is possible to avoid the dependency in γ by multiplying C by $e^{\gamma_0 T}$. So we can rewrite (5)-(6) under the more pleasant form :

$$\|u\|^2_{L^2(\Omega_T)} + \|u_{|x_d=0}\|^2_{H^{-\frac{1}{2}}(\omega_T)} \leq C_T\left(\|f\|^2_{L^2_{x_d}(H^s(\omega_T))} + \|g\|^2_{H^{\frac{1}{2}}(\omega_T)}\right),$$

with $s = 0$ for (5) and $s = \frac{1}{2}$ for (6), f, g, u zero for $t < 0$.
Our first result is that one can make a geometric optics expansion if $\underline{\zeta}$ satisfies assumption 2.4 or 2.5. The oscillating source terms f^ε, g^ε are made precise in (9) under assumption 2.4 and (9) under assumption (24).

Theorem 2.3 *Under assumptions 2.1-2.2-2.3-**2.4**, one can find a geometric optics expansion $(U_n)_{n\in\mathbb{N}}$ of the ibvp (1) which satisfy equations (11) (see below). Moreover, setting $u^\varepsilon_{N_0,app}$ the truncated geometric optics expansion defined by :*

$$u^\varepsilon_{N_0,app} = \sum_{n=0}^{N_0} e^{i\frac{\varphi}{\varepsilon}}\varepsilon^n U_n, \tag{7}$$

then the error beetween $u^\varepsilon_{N_0,app}$ and the exact solution u^ε of (1) is an $O(\varepsilon^{N_0+\frac{3}{2}})$ in H^∞_ε (see 4.1).
*Under assumptions 2.1-2.2-2.3-**2.5**, one can find a geometric optics expansion $(U_n)_{n\in\mathbb{N}}$ of the ibvp (1) which satisfy equations (26)-(27) (see below). Such that the error beetween $u^\varepsilon_{N_0,app}$ and the exact solution u^ε of (1) is an $O(\varepsilon^{N_0+1})$ in H^∞_ε (see 4.1).*

The two following theorems show that energy estimates (5) and (6) are sharp :

Theorem 2.4 *Under assumptions 2.1-2.2-2.3-**2.4**, let $s > 0$. We assume that for all f in $L^2(\Omega_T)$, $g \in H^s(\omega_T)$ vanishing for $t \leq 0$, there is a unique solution u in $L^2(\Omega_T)$ of the ibvp (1) with the energy estimate :*

$$\|u\|^2_{L^2(\Omega_T)} \leq C_T \left(\|f\|^2_{L^2(\Omega_T)} + \|g\|^2_{H^s(\omega_T)} \right),$$

then necessarily $s \geq \frac{1}{2}$.

Theorem 2.5 *Under assumptions 2.1-2.2-2.3-**2.5**-2.6, let $s_1, s_2 > 0$. We assume that for all f in $L^2_{x_d}(H^{s_1}(\omega_T))$, $g \in H^{s_2}(\omega_T)$ vanishing for $t \leq 0$, there is a unique solution u in $L^2(\Omega_T)$ of the ibvp (1) with the energy estimate :*

$$\|u\|^2_{L^2(\Omega_T)} \leq C_T \left(\|f\|^2_{L^2_{x_d}(H^{s_1}(\omega_T))} + \|g\|^2_{H^{s_2}(\omega_T)} \right), \tag{8}$$

then necessarily $s_1 \geq \frac{1}{2}$ and $s_2 \geq \frac{1}{2}$.

Theorem 2.5 indicates that the existence of hyperbolic outgoing modes has a serious impact on the stability properties of (1). In particular, strong well-possedness of (1) for $g \equiv 0$, as considered in [[3], chapter 7] can not occur if such modes are present. This result gives a rigorous justification to the discussion in [[3] pages 205-207]. We refer to section 5 for further comments and consequences.

3 Geometric optics expansions.

Let $\varphi(t, x') = \underline{\tau} t + \underline{\eta}.x'$, with $(\underline{\tau}, \underline{\eta})$ a frequency that satisfies assumption 2.4 or assumption 2.5.

3.1 Geometric optics expansions under assumption 2.4.

This section gives a rigorous construction of geometric optics expansions when UKL fails in the elliptic area. It is a generalization of Marcou's work in the case of a non-homogenous boundary condition. Indeed, in this paper we are interested in the influence of a non-homogenous boundary condition on the energy estimate.
Moreover, the case of an elliptic frequency will be a good preparatory work for the case of a mixed frequency in the next paragraph.

Remark Hereinafter as no confusion is possible we will set $\Pi^e_\pm = \Pi_\pm$.

From now to the end of the paragraph, we will deal with source terms f^ε and g^ε of the form,

$$\begin{aligned} f^\varepsilon(t, x) &= e^{i\frac{\varphi}{\varepsilon}} f^{ev}\left(t, x, \frac{x_d}{\varepsilon}\right), \\ g^\varepsilon(t, x') &= \varepsilon e^{i\frac{\varphi}{\varepsilon}} g(t, x'), \end{aligned} \tag{9}$$

with amplitudes f^{ev} in P_{ev} and g in $H^{+\infty}(\omega_T)$,both zero for negative times.
We also take an ansatz of the form,

$$u^\varepsilon(t,x) = \sum_{n\geq 0} \varepsilon^n e^{i\frac{\varphi}{\varepsilon}} U_n\left(t,x,\frac{x_d}{\varepsilon}\right), \tag{10}$$

where for all $n>0$, U_n is an element of P_{ev}.
Taking such ansatz allows us to treat elliptic modes in one piece as in Lescarret's work [11]. Thus, in comparison to Williams's method in [17] see also [8] it is unnecesary to assume that the stable part of $\mathcal{A}(\underline{\zeta})$ is diagonalizable, moreover it does not require to solve in an approximate way transport equations with complex coefficients.
Plugging the ansatz (10) in the ibvp (1), the following cascade of equations appears :

$$\left\{\begin{array}{l} L(\partial_X)U_0 = 0, \\ L(\partial_X)U_n + L(\partial)U_{n-1} = \delta_{n,1}f, \forall n\geq 1, \\ BU_{n_{|x_d=X_d=0}} = \delta_{n,1}g, \forall n, \\ U_{n_{|t\leq 0}} = 0, \forall n \end{array}\right. \tag{11}$$

where $\delta_{n,p}$ is the Kronecker's symbol and :

$$L(\partial_X) := A_d\left(\partial_{X_d} - \mathcal{A}(\underline{\zeta})\right).$$

Lemma 3.1 *Let*

$$\mathbb{P}_{ev}U(X_d) := e^{X_d\mathcal{A}(\underline{\zeta})}\Pi_- U(0), \tag{12}$$

$$\mathbb{Q}_{ev}F(X_d) := \int_0^{X_d} e^{(X_d-s)\mathcal{A}(\underline{\zeta})}\Pi_- A_d^{-1}F(s)ds - \int_{X_d}^{+\infty} e^{(X_d-s)\mathcal{A}(\underline{\zeta})}\Pi_+ A_d^{-1}F(s)ds. \tag{13}$$

Then for all F in P_{ev}, the equation

$$L(\partial_X)U = F,$$

has a solution U in P_{ev}. Moreover, one can write any such solution U under the following form :

$$U = \mathbb{P}_{ev}U + \mathbb{Q}_{ev}F.$$

proof : $\mathbb{P}_{ev}$ and $\mathbb{Q}_{ev}$ defined in (12)-(13) are explicitly given by Duhamel's formula.

Using lemma 3.1, the cascade of equations (11) can be rewritten equivalently as follows :

$$\left\{\begin{array}{l} \mathbb{P}_{ev}U_0 = U_0, \\ BU_{0_{|x_d=X_d=0}} = 0\ , \\ U_{0_{|t\leq 0}} = 0 \end{array}\right. \tag{14}$$

and for higher order terms :

$$\left\{\begin{array}{l}(I-\mathbb{P}_{ev})U_n=\mathbb{Q}_{ev}(\delta_{n,1}f^{ev}-L(\partial_x)U_{n-1})\ ,\\ BU_{n_{|x_d=X_d=0}}=\delta_{n,1}g\ ,\\ U_{n_{|t\leq 0}}=0\ ,\end{array}\right. \qquad \forall n\geq 1. \tag{15}$$

3.1.1 Determining the leading order term U_0.

Using $\mathbb{P}_{ev}U_0=U_0$ in the boundary condition of (14) one has to solve the following system for the principal term :

$$\left\{\begin{array}{l}\mathbb{P}_{ev}U_0=U_0\ ,\\ B\Pi_-U_0(t,x',0,0)=0\ ,\\ U_{0_{|t\leq 0}}=0\ .\end{array}\right. \tag{16}$$

Thanks to equation (12), the first equation implies that in order to know U_0 it is sufficient to know its trace on $\{X_d=0\}$. Unfortunately, the boundary condition does not determine this trace but a double trace on $\{X_d=x_d=0\}$. Moreover, this equation can not be solved easily due to the degeneracy of UKL on $E_-(\underline{\zeta})$. We will start by determining the double trace $U_0(t,x',0,0)$. Let

$$\Pi_-U_0(t,x',0,0)=\alpha_0(t,x')e, \tag{17}$$

with e given in definition 2.5, α_0 a scalar function.
The determination of α_0 follows Marcou's method described in [12], which is here adapted to a non homogenous boundary condition.
Equations satisfied by the term U_1 are (see (15)) :

$$\left\{\begin{array}{l}(I-\mathbb{P}_{ev})U_1=\mathbb{Q}_{ev}(f^{ev}-L(\partial_x)U_0)\ ,\\ BU_{1_{|x_d=X_d=0}}=g\ ,\\ U_{1_{|t\leq 0}}=0\ .\end{array}\right. \tag{18}$$

Injecting, the first equation written for $x_d=X_d=0$ in the boundary condition and after multiplying by the vector b (given in definition 2.5), we are led to :

$$b.B\mathbb{Q}_{ev}(f^{ev}-L(\partial_x)U_0)_{|x_d=X_d=0}=b.g\quad .$$

However by definition of $\mathbb{Q}_{ev}$ (see (13)), we can develop the left-hand side of the previous equation as follows :

$$b.B\mathbb{Q}_{ev}(f^{ev}-L(\partial_x)U_0)_{|x_d=X_d=0}=b.B\int_0^{+\infty}e^{-s\mathcal{A}(\underline{\zeta})}\Pi_+A_d^{-1}(-f^{ev}+L(\partial_x)U_0(s))ds.$$

We thus obtain

$$\begin{aligned}b.B\int_0^{+\infty}e^{-s\mathcal{A}(\underline{\zeta})}\Pi_+A_d^{-1}L(\partial_x)U_0(s)ds &= b.\left(g+B\int_0^{+\infty}e^{-s\mathcal{A}(\underline{\zeta})}\Pi_+A_d^{-1}f^{ev}(s)ds\right)\\ &:= \tilde{g}_0.\end{aligned}$$

Once more developing the left-hand side and using the equality (17) we have the final equation :

$$b.B \int_0^{+\infty} e^{-s\mathcal{A}(\underline{\zeta})} \Pi_+ A_d^{-1} L(\partial_x) U_0(s) ds = b.B I_t \partial_t \alpha_0 + \sum_{j=1}^{d-1} b.B I_j \partial_j \alpha_0 \,,$$

where we set :

$$\left\{ \begin{array}{l} I_t = \int_0^{+\infty} e^{-s\mathcal{A}(\underline{\zeta})} \Pi_+ A_d^{-1} e^{s\mathcal{A}(\underline{\zeta})} e \, ds, \\ I_j = \int_0^{+\infty} e^{-s\mathcal{A}(\underline{\zeta})} \Pi_+ A_d^{-1} A_j e^{s\mathcal{A}(\underline{\zeta})} e \, ds. \end{array} \right. \tag{19}$$

So it is clear that α_0 satisfies the transport equation

$$\left(b.B I_t \partial_t + \sum_{j=1}^{d-1} b.B I_j \partial_j \right) \alpha_0 = \tilde{g}_0.$$

It does not seem obvious that the coefficients of this equation are real. So we can not conclude that this equation is well-posed yet. To do that, we will need to make the I_t, I_j more explicit which is possible thanks to the following lemma.

Lemma 3.2 *[Marcou] Set $I_t = I_0$, $\partial_{\eta_0} = \partial_t$ then for all $j \in \{0, ..., d-1\}$,*

$$I_j = \Pi_+ \partial_{\eta_j} \Pi_-(\underline{\zeta}) e,$$

where we use the notation $\eta_0 = \tau$.

proof : This proof is the same as in [[12], lemma 6.3] . We recall it here for the sake of completeness.
Let μ be an eigenvalue of $\mathcal{A}(\underline{\zeta})\Pi_-$, we have $Re\mu < 0$ and moreover $e^{s\mu}$ is an eigenvalue of $e^{s\mathcal{A}(\underline{\zeta})}\Pi_-$; applying Dunford's formula we can write :

$$e^{s\mathcal{A}(\underline{\zeta})}\Pi_- = \frac{1}{2i\pi} \int_{\Gamma_-} e^{sz} (\mathcal{A}(\underline{\zeta}) - z)^{-1} \, dz,$$

where Γ_- is a closed simple curve included in the complex half-plane with negative real part, surronding the eigenvalues of $\mathcal{A}(\underline{\zeta})\Pi_-$.
As $\Pi_- e = e$, we have

$$I_j = \frac{1}{2i\pi} \int_{\Gamma_-} \left(\int_0^{+\infty} e^{s(z - \mathcal{A}(\underline{\zeta}))} \Pi_+ \, ds \right) A_d^{-1} A_j (\mathcal{A}(\underline{\zeta}) - z)^{-1} \, dz \, e.$$

On the other hand we can explicitely compute

$$\begin{aligned} \int_0^{+\infty} e^{s(z-\mathcal{A}(\underline{\zeta}))} \Pi_+ \, ds &= \left[e^{s(z-\mathcal{A}(\underline{\zeta}))} \Pi_+ (z - \mathcal{A}(\underline{\zeta}))^{-1} \right]_0^{+\infty} \\ &= \Pi_+ (\mathcal{A}(\underline{\zeta}) - z)^{-1}. \end{aligned}$$

Setting $A_0 = I$ we have,

$$
\begin{aligned}
I_j &= \Pi_+ \left(\frac{1}{2i\pi} \int_{\Gamma_-} (\mathcal{A}(\underline{\zeta}) - z)^{-1} A_d^{-1} A_j (\mathcal{A}(\underline{\zeta}) - z)^{-1} dz \right) e \\
&= \Pi_+ \left(\frac{1}{2i\pi} \int_{\Gamma_-} \partial_{\eta_j} \left((\mathcal{A}(\underline{\zeta}) - z)^{-1} \right) dz \right) e \\
&= \Pi_+ \partial_{\eta_j} \Pi_-(\underline{\zeta}) e
\end{aligned}
$$

□

Thanks to the stability assumption 2.4 one can write in a neighborhood of $\underline{\zeta}$ the equality :

$$
Be(\zeta) = \beta(\zeta) \left(\gamma + i\theta(\zeta) \right) e(\zeta), \tag{20}
$$

with e, σ, β smooth such that

$$
\begin{cases}
e(\underline{\zeta}) = e \ , \\
\theta(\underline{\zeta}) = 0 \ ; \ \theta(\zeta) \in \mathbb{R} \ , \\
\beta(\underline{\zeta}) \neq 0 \ .
\end{cases}
$$

Using lemma 3.2, we also have in a neighborhood of $\underline{\zeta}$,

$$
b.BI = b.B\Pi_+(\underline{\zeta}) \partial \Pi_-(\underline{\zeta}) e.
$$

Differentiating the identity $\Pi_-(\zeta) e(\zeta) = e(\zeta)$ in a neighborhood of $\underline{\zeta}$ then composing on the left by $\Pi_+(\underline{\zeta})$ in order to avoid the second term we have :

$$
\Pi_+(\underline{\zeta}) \partial \Pi_-(\underline{\zeta}) e = \partial e(\underline{\zeta}).
$$

Now differentiating (20) in $\underline{\zeta}$ we obtain :

$$
B \partial e(\underline{\zeta}) = i\beta(\underline{\zeta}) \partial \theta(\underline{\zeta}) e.
$$

Therefore we have proved the following proposition :

Proposition 3.1 *The function α_0 defined in (17) is a solution of the real transport equation*

$$
\begin{cases}
i\beta(\underline{\zeta}) b.e \left(\partial_\tau \theta(\underline{\zeta}) \partial_t \alpha_0 + \nabla_\eta \theta(\underline{\zeta}) . \nabla_x \alpha_0 \right) = \tilde{g}_0 \ , \\
\alpha_{0|t \le 0} = 0 \ ,
\end{cases} \tag{21}
$$

thus α_0 is uniquely determined and it is nonzero if and only if $\tilde{g}_0$ is. Moreover α_0 has the same regularity as $\tilde{g}_0$ (ie $H^{+\infty}(\omega_T)$). By the equation (17), the same properties are also true for $\Pi_- U_0(t, x', 0, 0)$.

We readily observe that $\tilde{g}_0$ is nonzero if we choose a nonzero g and $f^{ev} \equiv 0$. This explains the scaling of g^ε in order to have an $O(1)$ solution u^ε in $L^\infty(\Omega_T)$. The double trace $\Pi_- U_0(t, x', 0, 0)$ is now fully determined. To conclude the construction of U_0 it is sufficient to note that in $U_0(t, x, X_d) = \mathbb{P}_{ev} U_0(t, x, X_d)$, x_d is a parameter. Consequently we can arbitrary lift the double trace $\Pi_- U_0(t,x',0,0)$ in a simple one $\Pi_- U_0(t,x,0)$. For instance, set as in [11] :

$$U_0(t,x,X_d) = \alpha_0(t,x')\chi(x_d)e^{X_d \mathcal{A}(\underline{\zeta})}e,$$

where $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ checking $\chi(0) = 1$.

Remark We stress that from now on the geometric optics expansion is not uniquely determined. Indeed it highly depends on the function χ.
But following Métivier's work [14] we can show that this dependency is only an error term of upper order in ε. Indeed, by Taylor's expansion in x_d :

$$U_0(t,x,X_d) = U_0(t,x',0,X_d) + x_d V(t,x,X_d),$$

where V is a function in P_{ev}, as well as $W := X_d V$. For $X_d = \frac{x_d}{\varepsilon}$:

$$U_0(t,x,X_d) = U_0(t,x',0,\frac{x_d}{\varepsilon}) + \varepsilon W(t,x,\frac{x_d}{\varepsilon}).$$

Consequently the difference beetwen two functions U_0 lifted by two different functions χ is at most a term of order one in ε.

3.1.2 Second term in the WKB expansion.

The construction of the second term of the expansion is very similar to that of the first one. Indeed the only modifications come from the fact that U_1 is not polarized so that we have to consider the unpolarized part and that there is a little change in the source term of the transport equation (21).
Let us recall that the second term satisfies equation (15). But thanks to the previous section from now U_0 is known, so the first equation of (15) allows to determine the unpolarized part of U_1.
Thus to complete the construct of U_1 it is sufficient to determine is polarized part that is to say $\mathbb{P}_{ev}U_1$. But reiterating the arguments of the previous section, to do that it is sufficient to know the trace $\Pi_- U_1(t,x,0)$.
Let us start by determining the double trace $\Pi_- U_1(t,x',0,0)$.
Plugging this double trace in the boundary condition of (15) we obtain :

$$B\Pi_- U_1(t,x',0,0) = g - \mathbb{Q}_{ev}(f^{ev} - L(\partial_x)U_0)_{|x_d = X_d = 0}. \tag{22}$$

As noticed in the introduction of this paragraph there is a little difference with the first term. Indeed, "a priori" the left side of the previous equation is not zero so we can not deduce that $\Pi_- U_1(t,x',0,0)$ is in $kerB \cap E_-(\underline{\zeta})$.
So let us decompose,

$$\Pi_- U_1(t,x',0,0) = \alpha_1(t,x')e + v_1(t,x'), \tag{23}$$

where α_1 is a scalar function and $v_1(t,x')$ is in $\check{E}_-(\underline{\zeta})$ (notation of definition 2.5).
The boundary condition (22) can now be rewritten as follows :

$$v_1(t,x') = B^{-1}_{|\check{E}_-(\underline{\zeta})}\left(g - \mathbb{Q}_{ev}(f^{ev} - L(\partial_x)U_0)_{|x_d = X_d = 0}\right),$$

which determines in a unique way v_1 in $H^{+\infty}(\omega_T)$, zero for negative times.
At this moment it is sufficient to determine α_1 to know the double trace $\Pi_- U_1(t,x',0,0)$. In order to do so, one can reiterate the construction described in paragraph 3.1.1 to exhibit a transport equation on α_1. We end up with :

Proposition 3.2 *The function α_1 defined in (23) is solution of the transport equation*

$$\left\{ \begin{array}{l} i\beta(\underline{\zeta})b.e\left(\partial_\tau\theta(\underline{\zeta})\partial_t\alpha_1 + \nabla_\eta\theta(\underline{\zeta}).\nabla_x\alpha_1\right) = \tilde{g}_1 \ , \\ {\alpha_1}_{|t\leq 0} = 0 \ , \end{array} \right.$$

with $\tilde{g}_1 := -b.B\mathbb{Q}_{ev}L(\partial_x)\Big((I-\Pi_-){U_1}_{|x_d=X_d=0} + v_1\Big)$

Mimicking the end of paragraph 3.1.1 we have to extend $\Pi_- U_1(t,x',0,0)$ to all positive x_d. Thus we can set

$$U_1(t,x,X_d) = \chi(x_d)e^{X_d\mathcal{A}(\underline{\zeta})}\left(\alpha_1(t,x')e + v(t,x')\right) + \mathbb{Q}_{ev}(f^{ev} - L(\partial_x)U_0)(t,x_d,X_d).$$

3.1.3 Higher order terms and summary.

It is clear that the process of the previous sections allows us to determine the $(n+1)^{th}$ term if we already know the terms uppon the n^{th}. Indeed let us summarize this method :
$\diamond$ If we know the n^{th} term, equation (15) determines the unpolarized part of the $(n+1)^{th}$ term.
$\diamond$ (12) implies that to know the polarized part it is sufficient to know $\Pi_- U_{n+1}(t,x,0)$. We begin by computing the double trace $\Pi_- U_{n+1}(t,x',0,0)$.
$\diamond$ To do so, we decompose $\Pi_- U_{n+1}(t,x',0,0)$ in a component on $kerB$ which depends on a scalar function α_{n+1} and a component on $\check{E}_-(\underline{\zeta})$ which is determined algebraically by the boundary condition.
$\diamond$ For α_{n+1} we write the equations for the $(n+2)^{th}$ term and Marcou's work allows to show that α_{n+1} is solution of a real transport equation. So at this stage $\Pi_- U_{n+1}(t,x',0,0)$ is known.
$\diamond$ To conclude it is sufficient to lift the double trace $\Pi_- U_{n+1}(t,x',0,0)$ in a "simple" trace $\Pi_- U_{n+1}(t,x,0)$.
Thus we have shown the following proposition :

Proposition 3.3 *Under assumptions 2.1-2.2-2.3-**2.4**, one can find a geometric optics expansion $(U_n)_{n\in\mathbb{N}}$ of the ibvp (1) which satisfy equations (11), where all U_n belong to P_{ev}.*

3.2 Geometric optics expansion under assumption 2.5.

From now on the frequency $\underline{\zeta}$ will satisfy assumption 2.5 raher than assumption 2.4. So we will have to include highly oscillating terms in the geometric optics expansion. The philosophy of the construction will be the same but we will have

to be more careful about the resolution of boundary conditions.
We choose source terms f^ε and g^ε under the form,

$$\begin{aligned} f^\varepsilon(t,x,X_d) &= e^{i\frac{\varphi}{\varepsilon}}\left(\varepsilon f^{os}(t,x,X_d) + f^{ev}(t,x,X_d)\right), \\ g^\varepsilon(t,x') &= \varepsilon e^{i\frac{\varphi}{\varepsilon}} g(t,x'), \end{aligned} \tag{24}$$

with $f^{os} = \sum_{m=1}^{M} e^{i\underline{\omega}_m X_d} f_m^{os}$ in P_{os}, f^{ev} in P_{ev} and g in $H^{+\infty}(\omega_T)$, both zero for the negative times.
Let us consider the ansatz

$$u^\varepsilon(t,x) = \sum_{n\geq 0} \varepsilon^n e^{i\frac{\varphi}{\varepsilon}} U_n\left(t,x,\frac{x_d}{\varepsilon}\right), \tag{25}$$

where for all integer n, $U_n \in P$ is decomposed as

$$U_n(t,x,X_d) := U_n^{os} + U_n^{ev} = \sum_{m=1}^{M} e^{i\underline{\omega}_m X_d} u_{n,m} + U_n^{ev}.$$

Injecting the ansatz (25) in the ibvp (1) we obtain the cascade of equations :

$$\begin{cases} L(\partial_X)U_0 = 0, \\ BU_{0_{|x_d=X_d=0}} = 0, \\ U_{0_{|t\leq 0}} = 0\ , \end{cases} \tag{26}$$

for the principal term. For higher order terms :

$$\begin{cases} L(\partial_X)U_n + L(\partial_x)U_{n-1} = \delta_{n,1} f^{ev} + \delta_{n,2} f^{os}, \\ BU_{n_{|x_d=X_d=0}} = \delta_{n,1} g\ , & \forall n \geq 1. \\ U_{n_{|t\leq 0}} = 0\ , \end{cases} \tag{27}$$

The following lemma which is just a generalization of lemma 3.1, gives us a solution of the equation $L(\partial_X)U = F$ with F in P.

Lemma 3.3 *[11] There are projectors $\mathbb{P}$, $\mathbb{P}^i$, a partial inverse $\mathbb{Q}$ such that for all profiles $F \in P$, the equation :*

$$L(\partial_X)U = F \tag{28}$$

admits a solution U in P if the following compatibility condition $\mathbb{P}^i F = 0$ is satisfied. In that case, this solution U can be written :

$$U = \mathbb{Q}F + \mathbb{P}U.$$

In particular on P_{ev}, $\mathbb{P} = \mathbb{P}_{ev}$, $\mathbb{Q} = \mathbb{Q}_{ev}$ and $\mathbb{P}^i$ is zero. On P_{os}, $\mathbb{P}^i = \mathbb{P}^i_{os} = \mathbb{P}_{os}$.

We refer to [11] for a complete proof but since it will be useful we give below the expression of $\mathbb{P}_{os}$, and $\mathbb{Q}_{os}$:

$$\mathbb{P}_{os} V = \mathbb{P}^i_{os} V = \sum_{m=1}^{M} e^{iX_d \underline{\omega}_m} \Pi_m V_m, \tag{29}$$

$$\mathbb{Q}_{os} V = \sum_{m=1}^{M} e^{iX_d \underline{\omega}_m} Q_m V_m,$$

where $V = \sum_{m=1}^{M} e^{i\underline{\omega}_m X_d} V_m$.
Using lemma 3.3, (26) and (27) can be rewritten as follows :

$$\begin{cases} \mathbb{P} U_0 = U_0, \\ \mathbb{P}^i L(\partial_x) \mathbb{P} U_0 = 0, \\ B U_{0_{|x_d = X_d = 0}} = 0 \ , \\ U_{0_{|t \le 0}} = 0 \ . \end{cases} \tag{30}$$

and

$$\begin{cases} (I - \mathbb{P}) U_n = \mathbb{Q}(\delta_{n,2} f^{os} + \delta_{n,1} f^{ev} - L(\partial_x) U_{n-1}) \ , \\ \mathbb{P}^i L(\partial_x) U_n = \mathbb{P}^i \delta_{n,1} f^{os} + \mathbb{P}^i \delta_{n,1} f^{ev} \ , \\ B U_{n_{|x_d = X_d = 0}} = \delta_{n,1} g \ , \\ U_{n_{|t \le 0}} = 0 \ . \end{cases} \quad \forall n \ge 1. \tag{31}$$

The first equations in (30)-(31) are due to the particular form of the solution of (28), and the second come from the compatibility condition $\mathbb{P}^i F = 0$.
We see that the first term U_0 has to satisfy the conditions (30) of polarization and propagation together with some boundary conditions. Decomposing on the profiles spaces P_{ev} and P_{os} it follows:

$$\begin{cases} \mathbb{P}_{ev} U_0^{ev} = U_0^{ev} \ , \\ \mathbb{P}_{os} U_0^{os} = U_0^{os} \ , \\ \mathbb{P}^i_{os} L(\partial_x) \mathbb{P}_{os} U_0^{os} = 0, \\ \mathbb{P}^i_{ev} L(\partial_x) \mathbb{P}_{ev} U_0^{ev} = 0, \\ B \left[U_0^{os} + U^{ev}_{0_{|X_d = 0}} \right]_{|x_d = 0} = 0 \ , \\ U^{os}_{0_{|t \le 0}} = U^{ev}_{0_{|t \le 0}} = 0 \ . \end{cases} \tag{32}$$

However, thanks to the relation $\mathbb{P}^i_{ev} = 0$ given by lemma 3.3 the fourth equation of (32) is a tautology. Since this lemma also gives the relation $\mathbb{P}^i_{os} = \mathbb{P}_{os}$ one can reformulate the third equation of (32) :

$$\mathbb{P}_{os} L(\partial_x) \mathbb{P}_{os} U_0^{os} = 0.$$

So we see that the evolution equations on the oscillating terms and on the evanescent terms are just linked by the boundary condition. Consequently one can hope to solve the evolution equations independently for the oscillating and the evanescent terms. Let us start by determining the principal oscillating term.

3.2.1 Leading order oscillating term.

Thanks to the decomposition given in lemma 2.1 and (29) we will identify each term in the expansion of U_0^{os} :

$$\left\{\begin{array}{l}\Pi_m u_{0,m} = u_{0,m}\ ,\\ \Pi_m L(\partial_x) u_{0,m} = 0\ ,\\ B\left[\sum_{m=1}^{M} u_{0,m} + U^{ev}_{0_{|X_d=0}}\right]_{|x_d=0} = 0\ ,\\ u_{0,m_{|t\leq 0}} = 0\ ,\end{array}\right. \quad \forall m\in\{1,M\}\,. \tag{33}$$

Let us recall the classical lemma due to Lax :

Lemma 3.4 *[10] Under assumption 2.1, the following equality is true*

$$\Pi_m L(\partial_x)\Pi_m u_{0,m} = (\partial_t + v_m.\nabla_x) u_{0,m},$$

where $v_m = \nabla\lambda_{k_m}(\underline{\eta},\underline{\omega}_m)$ *is the group velocity associated to* λ_{k_m} *defined in (4)* .

Distinguish two cases according to the set in which m lies.
First case : $m\in\mathcal{NC}$
The group velocity v_m is outgoing thus the transport equation associated to (33) is :

$$\left\{\begin{array}{l}(\partial_t + v_m.\nabla_x) u_{0,m} = 0\ ,\\ u_{0,m_{|t\leq 0}} = 0\ ,\end{array}\right.$$

and can be explicitely solved by integration along the characteristics. Consequently $u_{0,m}$ as solution of a homogeneous transport equation is zero and the same remains true for its trace on $\{x_d = 0\}$.
Second case : $m\in\mathcal{C}$
Conversely when m is causal the phenomenon is incoming so the transport equation has to take the boundary condition into account; thus it reads :

$$\left\{\begin{array}{l}(\partial_t + v_m.\nabla_x) u_{0,m} = 0\ ,\\ B\left[\sum_{m\in\mathcal{C}} \Pi_m u_{0,m} + (\mathbb{P}_{ev} U_0^{ev})_{|X_d=0}\right]_{|x_d=0} = -B\sum_{m\in\mathcal{NC}} \Pi_m u_{0,m} = 0\ ,\\ u_{0,m_{|t\leq 0}} = 0\ .\end{array}\right.$$

Focusing on the left-hand side of the boundary condition, we obtain :

$$\sum_{m\in\mathcal{C}} \Pi_m u_{0,m_{|x_d=0}} + \Pi_-^e U_0^{ev}(t,x',0,0) \in vect\{e.\}$$

So in order to satisfy the boundary condition, we need

$$\left\{\begin{array}{l}\Pi_m u_{0,m_{|x_d=0}} = 0\ ,\ \forall m\in\mathcal{C}\\ \Pi_-^e U_0^{ev}(t,x',0,0) = \alpha_0(t,x') e\ ,\end{array}\right. \tag{34}$$

where α_0 is a scalar function.
Consequently the transport equation for a causal term reads :

$$\begin{cases} (\partial_t + v_m.\nabla_x) u_{0,m} = 0 \ , \\ u_{0,m_{|x_d=0}} = 0 \ , \\ u_{0,m_{|t\leq 0}} = 0 \ ; \end{cases}$$

one more time, one can solve it explicitly and show that $u_{0,m}$ and the trace on $\{x_d = 0\}$ are zero.

3.2.2 Principal term for the evanescent part.

For the principal evanescent term we have to solve the same cascade of equations as in the previous paragraph (see (16)), that is to say :

$$\begin{cases} \mathbb{P}_{ev} U_0^{ev} = U_0^{ev} \ , \\ B\Pi_-^e U_0^{ev}(t, x', 0, 0) = 0 \ , \\ U^{ev}_{0_{|t\leq 0}} = 0 \ . \end{cases} \tag{35}$$

So we will try to apply the method described in paragraph 3.1.1 up to a few modifications.
As before, equations for the higher order evanescent term U_1^{ev} read :

$$\begin{cases} (I - \mathbb{P}_{ev}) U_1^{ev} = \mathbb{Q}_{ev}(f^{ev} - L(\partial_x) U_0^{ev}), \\ B\left[U_1^{os} + U^{ev}_{1_{|X_d=0}}\right]_{|x_d=0} = g, \\ U^{ev}_{1_{|t\leq 0}} = 0. \end{cases} \tag{36}$$

The only change is that now oscillating terms of order one appear in the boundary condition. Since we do not know those terms, Marcou's method can not be applied at this stage.
However, in paragraph 3.2.1 we managed to determine oscillating terms without any consideration on the evanescent ones, so we will try to do that once more.

3.2.3 Second oscillating terms in the WKB expansion.

Using the cascade of equations (31) for the oscillating part, we obtain

$$\begin{cases} (I - \mathbb{P}_{os}) U_1^{os} = -\mathbb{Q}_{os}(L(\partial_x) U_0^{os}), \\ \mathbb{P}_{os} L(\partial_x) U_1^{os} = f^{os}, \\ BU_{1_{|x_d = X_d = 0}} = g, \\ U^{os}_{1_{|t\leq 0}} = 0 \ . \end{cases} \tag{37}$$

Since from now we know that U_0^{os} is zero, the first equation allows to determine the unpolarized part of U_1^{os} and it is zero. So it is sufficient to determine the polarized part of U_1^{os}, to do that combining the first and the second equations of (37) it follows :

$$\mathbb{P}_{os} L(\partial_x) \mathbb{P}_{os} U_1^{os} = f^{os}.$$

Reiterating the arguments of paragraph 3.2.1, we exhibit the following transport equation on $\Pi_m u_{1,m}$:

$$\left\{\begin{array}{l}(\partial_t + v_m.\nabla_x)\Pi_m u_{1,m} = f_m^{os}\ ,\\ \Pi_m u_{1,m_{|t\leq 0}} = 0\ ,\end{array}\right. \quad , m \in \mathcal{NC}. \tag{38}$$

And for $m \in \mathcal{C}$:

$$\left\{\begin{array}{l}(\partial_t + v_m.\nabla_x)\Pi_m u_{1,m} = f_m^{os}\ ,\\ B\left(\sum_{m\in\mathcal{C}} \Pi_m u_{1,m_{|x_d=0}} + U_1^{ev}(t,x',0,0)\right) = g - B\sum_{m\in\mathcal{NC}} \Pi_m u_{1,m_{|x_d=0}}\\ \Pi_m u_{1,m_{|t\leq 0}} = 0\ ,\end{array}\right. \tag{39}$$

(38) is a transport equation which determines in a unique way $\Pi_m u_{1,m} = u_{1,m}$ and its trace on $\{x_d = 0\}$ for all m in $\mathcal{NC}$. We stress that $u_{1,m}$ is not zero unless the source terms meet specific requirements.
We have to be more careful with causal terms. Indeed, as said at the end of paragraph 3.2.2, evanescent and oscillating terms both appear in the boundary condition.
Let us decompose the evanescent term in its polarized and unpolarized parts. The boundary condition in (39) now reads :

$$\begin{aligned} B\left(\sum_{m\in\mathcal{C}} \Pi_m u_{1,m_{|x_d=0}} + \Pi_-^e U_1^{ev}(t,x',0,0)\right) &= g - B\sum_{m\in\mathcal{NC}} \Pi_m u_{1,m_{|x_d=0}}\\ &- B\mathbb{Q}_{ev}(f^{ev} - L(\partial_x)U_0^{ev}),\end{aligned} \tag{40}$$

where we used :

$$(I - \mathbb{P}_{ev})U_1^{ev} = \mathbb{Q}_{ev}(f^{ev} - L(\partial_x)U_0^{ev}).$$

The main point is that one more time the vector

$$\sum_{m\in\mathcal{C}} \Pi_m u_{1,m_{|x_d=0}} + \Pi_-^e U_1^{ev}(t,x',0,0),$$

which appears on the left-hand side of (40) is an element of $E_-(\underline{\zeta})$.
Thus after multiplying by the vector b defined in definition 2.5 we obtain the following compatibility condition :

$$b.B\mathbb{Q}_{ev}(f^{ev} - L(\partial_x)U_0^{ev}) = b.\left(g - B\sum_{m\in\mathcal{NC}} \Pi_m u_{1,m_{|x_d=0}}\right). \tag{41}$$

Moreover thanks to the definition 2.5 one can find w_1 in $\breve{E}_-(\underline{\zeta})$ such that the following decomposition is satisfied :

$$\sum_{m\in\mathcal{C}} \Pi_m u_{1,m_{|x_d=0}} + \Pi_-^e U_1^{ev}(t,x',0,0) = \tilde{\alpha}_1(t,x')e + w_1(t,x'). \tag{42}$$

Applying B, (39) implies that w_1 is given by the formula :

$$w_1(t,x') = B^{-1}_{|\check{E}_-(\underline{\zeta})}\left(g - B\sum_{m\in\mathcal{NC}}\Pi_m u_{1,m_{|x_d=0}} - B\mathbb{Q}_{ev}(f^{ev} - L(\partial_x)U_0^{ev})\right), \tag{43}$$

showing that w_1 is known if and only if U_0^{ev} is determined.

3.2.4 End of the construction of the evanescent term.

The compatibility condition (41) will allow to apply Marcou's method (see 3.1.1) indeed it reads :

$$\begin{aligned} b.B\mathbb{Q}_{ev}(f^{ev} - L(\partial_x)U_0^{ev})_{|x_d=X_d=0} &= b.\left(g - B\sum_{m\in\mathcal{NC}} u_{1,m_{|x_d=0}}\right) \\ &:= \tilde{g}_0(t,x'), \end{aligned}$$

where all source terms on the right-hand side are known. So one can show the following proposition

Proposition 3.4 *The function α_0 defined in (34) is solution of the transport equation*

$$\begin{cases} i\beta(\underline{\zeta})b.e\left(\partial_\tau\theta(\underline{\zeta})\partial_t\alpha_0 + \nabla_\eta\theta(\underline{\zeta}).\nabla_x\alpha_0\right) = \tilde{g}_0 \ , \\ \alpha_{0_{|t\leq 0}} = 0 \ , \end{cases} \tag{44}$$

Thus α_0 is determined in a unique way. It is zero if and only if $\tilde{g}_0$ is zero, and it has the same regularity as $\tilde{g}_0$. Thanks to (34) it is the same for $\Pi_-^e U_0^{ev}(t,x',0,0)$.

Reiterating the end of the analysis of paragraph 3.1.1, it is sufficient to lift $\Pi_-^e U_0^{ev}(t,x',0,0)$ to obtain a solution of (35). A typical example is

$$U_0^{ev}(t,x,X_d) = \alpha_0(t,x')\chi(x_d)e^{X_d\mathcal{A}(\underline{\zeta})}e, \tag{45}$$

with $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\chi(0) = 1$.

3.2.5 End of the construction for causal terms.

After the choice of χ, the principal evanescent term is determined. So, it is the same for w_1 defined in (42) and it is explictly given by the formula (43).
However, w_1 is in $\check{E}_-(\underline{\zeta})$ so it can be decomposed as follows :

$$w_1 = v_1^h + v_1^e,$$

with v_1^h an element of $E_-^h(\underline{\zeta})$, v_1^e in a supplementary space of $E_-^h(\underline{\zeta})$ in $\check{E}_-(\underline{\zeta})$. We emphasize that v_1^e and v_1^h are known since w_1 is known from equation (43). By identification on those spaces we obtain the two equations :

$$\begin{cases} \sum_{m\in\mathcal{C}} u_{1,m_{|x_d=0}} = v_1^h. \\ \Pi_-^e U_1^{ev}(t,x',0,0) = \alpha_1(t,x')e + v_1^e(t,x') \end{cases} .$$

Consequently the transport equation (39) can be rewritten as follows :

$$\left\{ \begin{array}{l} (\partial_t + v_m.\nabla_x)\Pi_m u_{1,m} = f_m^{os} \ , \\ \sum_{m\in\mathcal{C}} \Pi_m u_{1,m_{|x_d=0}} = v_1^h \\ \Pi_m u_{1,m_{|t\leq 0}} = 0 \ , \end{array} \right. \quad , m \in \mathcal{C}.$$

and it is now possible to determine $u_{1,m}$ for all causal m, by integration along the characteristics.

3.2.6 Second evanescent term.

The method to build the second evanescent term is similar to the method for the first one.
Taking equation (36), since U_0^{ev} is now known it is sufficient to determine the polarized part that is to say $\Pi_-^e U_1^{ev}(t, x, 0)$.
Decomposing as follows :

$$\Pi_-^e U_1(t, x', 0, 0) = \alpha_1(t, x')e + v_1^e(t, x'),$$

where $v_1^e(t, x') \in \breve{E}_-(\underline{\zeta})$ is given by the boundary condition.
Consequently it is sufficient to know α_1. To do that we will need to deal with equations on the third evanescent term and since it contains oscillating terms we will need to know them as well. As before, noncausal terms are easy to determine since they satisfy outgoing transport equations. For causal terms we have to decompose the boundary condition for $n = 2$ as in sections 3.2.3 and 3.2.4, one can show that α_1 is the solution of the transport equation :

$$\left\{ \begin{array}{l} i\beta(\underline{\zeta})b.e\left(\partial_\tau\theta(\underline{\zeta})\partial_t\alpha_1 + \nabla_\eta\theta(\underline{\zeta}).\nabla_x\alpha_1\right) = \tilde{g}_1 \ , \\ \alpha_{1_{|t\leq 0}} = 0 \ . \end{array} \right.$$

So it is now sufficient to extend the double trace $\Pi^- U_1^{ev}(t, x', 0, 0)$ for the positive x_d. For instance, we can set :

$$U_1^{ev}(t, x, X_d) = \chi(x_d)e^{X_d\mathcal{A}(\underline{\zeta})}\left[\alpha_1(t, x')e + v(t, x')\right] + \mathbb{Q}_{ev}(f^{ev} - L(\partial_x)U_0^{ev})(t, x, X_d).$$

3.2.7 Higher order terms.

Let us summarize the construction of the higher order terms.
⋄ The first equation of (31) gives the unpolarized parts of U_{n+1}^{os} and U_{n+1}^{ev}. So one more time it is sufficient to determine the polarized part.
⋄ For noncausal oscillating terms , the second equation of (31) and Lax's lemma lead to an outgoing transport equation :

$$\left\{ \begin{array}{l} (\partial_t + v_m.\nabla_x)\Pi_m u_{n+1,m} = F_m \ , \\ \Pi_m u_{n+1,m_{|t\leq 0}} = -(I - \Pi_m)u_{n+1,m_{|t\leq 0}} = 0 \ , \end{array} \right.$$

where F_m is a function depending on $(I - \Pi_m)u_{n+1,m}$ and possibly on the oscillating source terms. Integrating along the characteristics, noncausal oscillating

terms are determined.
⋄ The boundary condition for the causal oscillating and the evanescent terms reads as follows :

$$\begin{aligned} B\left(\sum_{m\in\mathcal{C}}\Pi_m u_{n+1,m_{|x_d=0}}+\Pi_-^e U_{n+1}^{ev}\right) &= -B\left(\sum_{m\in\mathcal{NC}} u_{n+1,m_{|x_d=0}}\right.\\ &+ ((I-\mathbb{P}_{ev})U_{n+1}^{ev})_{|x_d=X_d=0}\\ &- \left.\sum_{m\in\mathcal{C}}(I-\Pi_m)u_{n+1,m_{|x_d=0}}\right). \end{aligned}$$

Decomposing as in (42) allows to determine $u_{n+1,m_{|x_d=0}}$ thanks to Lax's lemma.
⋄ To determine the transport coefficient on the boundary of U_{n+1}^{ev} we write the boundary condition for U_{n+2}^{ev} , after the determination of the noncausal oscillating terms of order $n+2$ it reads :

$$B\left(\sum_{m\in\mathcal{C}}\Pi_m u_{n+2,m_{|x_d=0}}+\Pi_-^e U_{n+2}^{ev}\right) = G+B(\mathbb{Q}_{ev}L(\partial_x)U^{ev}_{n+1_{|x_d=X_d=0}})$$

with G a function depending on the traces of $u_{n+2,m}$, $(I-\Pi_m)u_{n+2,m}$ and possibly of the source terms on the boundary. Composing by b gives a compatibility condition on $U^{ev}_{n+1_{|x_d=X_d=0}}$ and Marcou's method allows to write it as a transport equation which permits to determine α_{n+1}.
Thus we have shown the following proposition :

Proposition 3.5 *Under assumptions 2.1-2.2-2.3-**2.5**, one can find a geometric optics expansion $(U_n)_{n\in\mathbb{N}}$ of the ibvp (1) which satisfy equations (26)-(27).*

4 Proof of the main theorems.

4.1 Justification of the expansion.

The aim of this section is to show that the two geometric optics expansions built in previous sections converge to the exact solution of (1).
Since proofs are very similar in both cases we will just give it in the case of the geometric optics expansion under assumption 2.5.
We give the following definition :

Definition 4.1 *For all integer N and for all $(u^\varepsilon)_\varepsilon$ of $H^{+\infty}(\Omega_T)$, we say that $(u^\varepsilon)_\varepsilon$ is $O(\varepsilon^N)$ in $H_\varepsilon^{+\infty}(\Omega_T)$ if for all $\alpha\in\mathbb{N}^{d+1}$ there is a positive constant C_α such that*

$$\forall\varepsilon\in\left]0,1\right],\varepsilon^{|\alpha|}\|\partial^\alpha u^\varepsilon\|_{L^2(\Omega_T)}\leq C_\alpha\varepsilon^N.$$

We will also need the following proposition in view of justifying the geometric optics expansion :

Proposition 4.1 *Let U be a fonction in P_{ev} then $U(t,x,\frac{x_d}{\varepsilon})$ and $(L(\partial_x)U(t,x,X_d))_{|X_d=\frac{x_d}{\varepsilon}}$ are $O(\varepsilon^{\frac{1}{2}})$ in $L^2(\Omega_T)$.*

proof : Since U is in P_{ev}, there is δ such that $e^{\delta X_d}U(t,x,X_d)$ is in $H^\infty(\Omega_T\times\mathbb{R}_+)$ so $(x_d,X_d)\mapsto U(\cdot,\cdot,x_d,X_d)e^{\delta X_d}\in L^\infty_{x_d,X_d}(L^2(\omega_T))$. We have,

$$\begin{aligned}
\|U_{|X_d=\frac{x_d}{\varepsilon}}\|^2_{L^2(\Omega_T)} &= \int_0^{+\infty}\int_{]-\infty,T]\times\mathbb{R}^{d-1}} |e^{\delta\frac{x_d}{\varepsilon}}U(t,x,\frac{x_d}{\varepsilon})e^{-\delta\frac{x_d}{\varepsilon}}|^2 dtdx'dx_d\\
&= \int_0^{+\infty}\|e^{\delta\frac{x_d}{\varepsilon}}U(\cdot,\cdot,x_d,\frac{x_d}{\varepsilon})\|^2_{L^2(\omega_T)}e^{-2\delta\frac{x_d}{\varepsilon}}dx_d\\
&\leq \|Ue^{\delta\cdot}\|_{L^\infty_{x_d,X_d}(L^2(\omega_T))}\int_0^{+\infty}e^{-2\delta\frac{x_d}{\varepsilon}}dx_d,\\
&\leq C\varepsilon.
\end{aligned}$$

Let ∂_y be a differential operator, y in $\{t,x_1,...,x_d\}$ we have :

$$\|\left(\partial_y U(\cdot,\cdot,X_d)\right)_{|X_d=\frac{x_d}{\varepsilon}}\|^2_{L^2(\Omega_T)} = \int_0^{+\infty} e^{-2\delta\frac{x_d}{\varepsilon}}\|e^{\delta\frac{x_d}{\varepsilon}}\partial_y U(\cdot,\cdot,x_d,\frac{x_d}{\varepsilon})\|^2_{L^2(\omega_T)}dx_d,$$

as the assessment X_d equals $\frac{x_d}{\varepsilon}$ is made after the derivative.
Since $e^{\delta X_d}\partial_y U(t,x,X_d)$ is in $H^\infty(\Omega_T\times\mathbb{R}_+)$ we can work as for the first assertion to show that

$$\|\left(\partial_y U(\cdot,\cdot,X_d)\right)_{|X_d=\frac{x_d}{\varepsilon}}\|^2_{L^2(\Omega_T)}\leq C\varepsilon,$$

and we conclude by the triangle inequality.

□

We have to prove the following result to complete the proof of theorem 2.3.

Theorem 4.1 *Under assumptions 2.1-2.2-2.3-**2.5**, $u^{N_0}_{app}-u^\varepsilon$ is an $O(\varepsilon^{N_0+1})$ in $H^{+\infty}_\varepsilon(\Omega_T)$, where we recall that $u^{N_0}_{app}$ is defined in (7) .*

proof : The remainder $u^{N_0+2}_{app}-u^\varepsilon$ satisfies the system :

$$\left\{\begin{array}{l}
L(\partial)\left(u^{N_0+2}_{app}-u^\varepsilon\right)=\varepsilon^{N_0+2}e^{i\frac{\varphi}{\varepsilon}}\left(L(\partial_x)U^{os}_{N_0+2}+L(\partial_x)U^{ev}_{N_0+2}\right)\\
B\left(u^{N_0+2}_{app}-u^\varepsilon\right)=0\\
u(t<0)=0
\end{array}\right.$$

Thanks to proposition 4.1 and by interpolation between spaces $L^2(\omega_T)$ and $H^1(\omega_T)$ one can show that for all index N, $L(\partial_x)U^{ev}_N$ is $O(1)$ in $L^2_{x_d}(H^{\frac{1}{2}}(\omega_T))$. The same kind of argument also shows that $L(\partial_x)U^{os}_N$ is $O(\varepsilon^{-\frac{1}{2}})$ in $L^2_{x_d}(H^{\frac{1}{2}}(\omega_T))$. Thus in terms of powers of ε the limiting term is the oscillating one.
Using the energy estimate (6), we obtain

$$\|u^{N_0+2}_{app}-u^\varepsilon\|_{L^2(\Omega_T)}\leq C\varepsilon^{N_0+\frac{3}{2}},$$

For all α, tangential derivatives $\partial^\alpha_{t,x'}$ are estimated directly by differentiation of the ibvp (1), so we have :

$$\|\partial^\alpha_{t,x'}\left(u^{N_0+2}_{app}-u^\varepsilon\right)\|_{L^2(\Omega_T)} \leq \varepsilon^{N_0+2}\|\partial^\alpha_{t,x'}\left(e^{i\frac{\varphi}{\varepsilon}}L(\partial_x)U_{N_0+2}\right)\|_{L^2(\Omega_T)}.$$

Thanks to Leibniz's formula it comes :

$$\begin{aligned}\|\partial^\alpha_{t,x'}\left(e^{i\frac{\varphi}{\varepsilon}}L(\partial_x)U_{N_0+2}\right)\|_{L^2(\Omega_T)} &\leq \frac{1}{\varepsilon^{|\alpha|}}\|L(\partial_x)U_{N_0+2}\|_{L^2(\Omega_T)}\\ &+ \|\partial^\alpha_{t,x'}\left(L(\partial_x)U_{N_0+2}\right)\|_{L^2(\Omega_T)}\\ &\leq \frac{C}{\varepsilon^{|\alpha|}}.\end{aligned}$$

Consequently,

$$\varepsilon^{|\alpha|}\|\partial^\alpha_{t,x'}\left(u^{N_0+2}_{app}-u^\varepsilon\right)\|_{L^2(\Omega_T)} \leq C\varepsilon^{N_0+\frac{3}{2}}.$$

Using the evolution equation, we can write the x_d-derivative of $u^{N_0+2}_{app}-u^\varepsilon$ as a linear combination the tangential derivatives. Thus the previous estimate still holds if we add derivatives in x_d. Consequently we have shown that $u^{N_0+2}_{app}-u^\varepsilon$ is $O(\varepsilon^{N_0+\frac{3}{2}})$ in $H^{+\infty}_\varepsilon(\Omega_T)$.
It is now easy to show that $u^{N_0+2}_{app}-u^{N_0}_{app}$ is $O(\varepsilon^{N_0+1})$ in $H^{+\infty}_\varepsilon(\Omega_T)$ and we conclude by the triangle inequality.

□

4.2 Optimality of energy estimate, proof of theorem 2.5

Once again since proofs are very similar in both cases we just describe here the proof of theorem 2.5. We will need the following lemma :

Lemma 4.1 *Let U_0 be as in (45), then $U_0(t,x,\frac{x_d}{\varepsilon})$ is $O(\varepsilon^{\frac{1}{2}})$ in $L^2(\Omega_T)$. This property is sharp in the sense that if there is $\kappa>0$ such that U_0 is an $O(\varepsilon^{\frac{1}{2}+\kappa})$ in $L^2(\Omega_T)$, then U_0 is zero in $L^2(\Omega_T)$.*

proof : The first assertion has already been proved in proposition 4.1.
To show the second assertion we will use the fact that for all $w\in\mathbb{C}^k$, $M\in\mathbf{M}_k(\mathbb{C})$ we have

$$|e^M w|^2 \geq e^{-2\|M\|}|w|^2.$$

So, if $v^\varepsilon = U_0\left(t,x,\frac{x_d}{\varepsilon}\right)$

$$\|v^\varepsilon\|_{L^2(\Omega_T)} \geq \|\alpha_0\|^2_{L^2}\left(\int_0^{+\infty}\chi(x_d)^2 e^{\frac{-2x_d}{\varepsilon}\|\mathcal{A}(\underline{\zeta})\|}|e|^2\,dx_d\right)^{\frac{1}{2}},$$

by change of variable $u=\frac{x_d}{\varepsilon}$ it comes

$$\begin{aligned}\|v^\varepsilon\|_{L^2(\Omega_T)} &\geq \varepsilon^{\frac{1}{2}}C_{\alpha_0,e}\left(\int_0^{+\infty}\chi(\varepsilon u)^2 e^{-2u\|\mathcal{A}(\underline{\zeta})\|}du\right)^{\frac{1}{2}},\\ &\geq C_{\alpha_0,e,\chi,\mathcal{A}(\underline{\zeta})}\varepsilon^{\frac{1}{2}}.\end{aligned}$$

$\square$

proof : (**theorem 2.5**) We argue by contradiction and assume $s_1 = \frac{1}{2} - \delta$, $\delta \in \left]0, \frac{1}{2}\right[$.
Let $g^\varepsilon \equiv 0$, and consider source terms

$$f^\varepsilon = \varepsilon e^{i\frac{\varphi}{\varepsilon}} f^{os}\left(t, x, \frac{x_d}{\varepsilon}\right),$$
$$f^{os}\left(t, x, \frac{x_d}{\varepsilon}\right) = \psi(t,x) e^{i\underline{\omega}_{m_0}\frac{x_d}{\varepsilon}} e_+^h, \tag{46}$$

where $e_+^h \in ker\mathcal{L}(i\underline{\zeta}, i\underline{\omega}_{m_0}) \setminus \{0\}$, which is possible from assumption 2.6.
So by interpolation between spaces $L^2(\omega_T)$ and $H^{\frac{1}{2}}(\Omega_T)$,

$$\|f^\varepsilon\|_{L^2_{x_d}(H^{s_1}(\omega_T))} \leq C\varepsilon^{\frac{1}{2}+\delta},$$

the energy estimate (8) allows to show that u^ε is $O(\varepsilon^{\frac{1}{2}+\delta})$ in $L^2(\Omega_T)$. Using the geometric optics expansion given by theorem 2.3 and thanks to the triangle inequality, one shows that $U_0 = U_0^{ev}$ the first term in the expansion is $O(\varepsilon^{\frac{1}{2}+\delta})$ in $L^2(\Omega_T)$.
But,

$$U_0 = U_0^{ev}\left(t, x, \frac{x_d}{\varepsilon}\right) = \alpha_0(t, x')\chi(x_d) e^{\frac{x_d}{\varepsilon}\mathcal{A}(\underline{\zeta})} e,$$

implies that for α_0 nonzero, U_0^{ev} is $O(\varepsilon^{\frac{1}{2}})$ in $L^2(\Omega_T)$. This result is sharp thanks to lemma 4.1. Therefore α_0 is zero. But, α_0 is solution of the transport equation (44) with :

$$\tilde{g}_0 = -b.B\left(\sum_{m\in\mathcal{NC}} u_{1,m_{|x_d=0}}\right).$$

The argument to conclude is the same as in [7]. For $m \in \mathcal{NC}$, $u_{1,m}$ is solution of the transport equation :

$$\begin{cases} (\partial_t + v_m.\nabla_x) u_{1,m} = \delta_{m,m_0}\psi(t,x) e_+^h \ , \\ u_{1,m_{|t\leq 0}} = 0 \ , \end{cases}$$

so the only nonzero amplitude is u_{1,m_0} and it is computable by integration along the characteristics.
Let us choose ψ in (46) such as the source term $-b.Bu_{1,m_{0_{|x_d=0}}}$ is nonzero. It implies that α_0 is nonzero and we have thus proved $s_1 \geq \frac{1}{2}$.
Once more we argue by contradiction so we suppose that $s_2 = \frac{1}{2} - \delta$, $\delta \in \left]0, \frac{1}{2}\right[$.
Let $f^\varepsilon \equiv 0$ and choose now

$$g^\varepsilon(t, x') := \varepsilon e^{i\frac{\varphi}{\varepsilon}} \psi(t, x') b,$$

with b as in definition 2.5 and $\psi \in \mathcal{C}_c^\infty$. Reiterating the same kind of interpolation arguments described in the first part of the proof and using the estimate

(8) one can show that u^ε is $O(\varepsilon^{\frac{1}{2}+\delta})$ in $L^2(\Omega_T)$. The geometric optics expansion tells us that

$$\|u^\varepsilon - e^{i\frac{\varphi}{\varepsilon}} U_0^{ev}\|_{L^2(\Omega_T)} \leq C\varepsilon.$$

So, U_0^{ev} is an $O(\varepsilon^{\frac{1}{2}+\delta})$ in $L^2(\Omega_T)$, consequently once again α_0 is zero. But, α_0 is the solution of the transport equation (44) with

$$\tilde{g}_0 = \psi |b|^2,$$

and we can choose ψ such that is not $\tilde{g}_0$ zero.So, α_0 as a solution of the nontrivial transport equation (44) is not zero which is the desired contradiction. So we proove that necessarily $s_2 \geq \frac{1}{2}$.

□

5 Consequences

5.1 Classification and well-posed homogeneous ibvp.

The first consequence of theorems 2.4-2.5 is to show rigorously the intuition in the beginning of [[3], chapter 7] and [15]. It says that the only well-posed homogeneous ibvp, that is ibvp of the form (1) with $g^\varepsilon \equiv 0$ and that satisfy the estimate :

$$\|u\|_{L^2(\Omega_T)} \leq C_T \|f\|_{L^2(\Omega_T)},$$

meet on of the following conditions :
$i)$ UKL is satisfied.
$ii)$ UKL fails in the elliptic region $\mathcal{E}$.
$iii)$ UKL fails in the glancing region $\mathcal{G}$ under some dimension restriction.
Indeed, if UKL fails in the mixed or in the hyperbolic area then necessarily there is a loss of at least half a derivative in the domain Ω_T, so the homogeneous ibvp is not well-posed.
If UKL fails in the elliptic region, thanks to theorem 2.4 the energy estimate is :

$$\|u\|^2_{L^2(\Omega_T)} + \|u_{|x_d=0}\|^2_{H^{-\frac{1}{2}}(\omega_T)} \leq C_T \left(\|f\|^2_{L^2(\Omega_T)} + \|g\|^2_{H^{\frac{1}{2}}(\omega_T)} \right),$$

and the only way to have strong well-posedness is that there is no loss of derivative on the boundary that is to say $g = 0$.
Moreover theorems 2.4-2.5 and the optimality of energy estimate when UKL fails in the hyperbolic region given by [7] allow the following classification of weakly well-posed ibvp when UKL fails outside the glancing area.
When UKL fails for an elliptic frequency then the energy estimate is :

$$\|u\|^2_{L^2(\Omega_T)} + \|u_{|x_d=0}\|^2_{H^{-\frac{1}{2}}(\omega_T)} \leq C_T \left(\|f\|^2_{L^2(\Omega_T)} + \|g\|^2_{H^{\frac{1}{2}}(\omega_T)} \right),$$

for a mixed frequency, we see that a loss of half a derivative appears in the interior of the domain :

$$\|u\|^2_{L^2(\Omega_T)} + \|u_{|x_d=0}\|^2_{H^{-\frac{1}{2}}(\omega_T)} \leq C_T \left(\|f\|^2_{L^2_{x_d}(H^{\frac{1}{2}}(\omega_T))} + \|g\|^2_{H^{\frac{1}{2}}(\omega_T)} \right).$$

At last, [7] show that the worst case occurs for hyperbolic frequencies for which we lose one derivative in the interior and one derivative on the boundary :

$$\|u\|^2_{L^2(\Omega_T)} + \|u_{|x_d=0}\|^2_{L^2(\omega_T)} \leq C_T \left(\|f\|^2_{L^2_{x_d}(H^1(\omega_T))} + \|g\|^2_{H^1(\omega_T)} \right). \tag{47}$$

5.2 Boundary conditions for linearized Euler equations.

Theorems 2.4-2.5 allow to describe areas where one can impose a maximal dissipative boundary condition for linearized Euler equations. Indeed this system is both symmetric and constantly hyperbolic so we can study the question of dissipative boundary conditions. Let us recall that the boundary conditions are maximal dissipative if the following is true :

$$\forall v \in kerB,\ \langle A_d v, v\rangle \leq 0,$$

It is known (see [3] chapter 3) that a symetrizable ibvp with a maximal dissipative boundary condition admits an energy estimate without loss of derivative in the interior.
Moreover for linearized Euler equations the elliptic area is empty so we have :

$$\Xi_0 = \mathcal{EH} \cup \mathcal{H} \cup \mathcal{G}.$$

Consequently thanks to theorem 2.5 and the optimality of the energy estimate (47) proved in [7], the only possibility to have a maximal dissipative boundary conditions that does not satisfy UKL is that the Lopatinskii determinant vanishes at a glancing point.
We deal with the linearized isentropic Euler equations in dimension 2 for an outgoing subsonic fluid. This correspond to the problem :

$$\begin{cases} \partial_t U + A_1\partial_1 U + A_2\partial_2 U = f\ , \\ BU_{|x_2=0} = 0\ , \\ U_{|t=0} = 0, \end{cases} \tag{48}$$

with :

$$A_1 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} M & 0 & -1 \\ 0 & M & 0 \\ -1 & 0 & M \end{bmatrix},$$

where $M \in]-1, 0[$ denotes the Mach number.
We suppose that B reads :

$$B = \begin{bmatrix} 1 & b_2 & b_3 \end{bmatrix},$$

with b_2, $b_3 \in \mathbb{R}$. We denote by $\mathcal{D}$ the set of $(b_2, b_3) \in \mathbb{R}^2$ such that B defines a maximal dissipative boundary condition and by $\tilde{\mathcal{G}}$ the set of $(b_2, b_3) \in \mathbb{R}^2$ such that UKL fails in the glancing region. $\tilde{\mathcal{G}}$ is easily computable (similar computations can be found in [[2],chapter 14]) and is given by :

$$\tilde{\mathcal{G}} = \left\{(b_2, b_3) \in \mathbb{R}^2 \setminus b_2 = \pm\frac{1 + Mb_3}{\sqrt{1 - M^2}}\right\} = \Gamma_+^g \cup \Gamma_-^g.$$

We introduce $\Delta(\sigma, \eta)$ a Lopatinskii determinant associated to the ibvp (48). The other possibilities of behaviour for a boundary condition (see [2]) are a bit more difficult to compute, we enumerate all the possible cases below

$$\begin{array}{rcl}
\Gamma^0 & := & \left\{(b_2, b_3) \in \mathbb{R}^2 \setminus \Delta(1, 0) = 0\right\} \\
& = & \left\{(b_2, b_3) \in \mathbb{R}^2 \setminus b_3 = 1\right\} \\
\Gamma^w & := & \left\{(b_2, b_3) \in \mathbb{R}^2 \setminus \exists \underline{\zeta} \in \mathcal{EH}, \Delta(\underline{\zeta}) = 0\right\} \\
& = & \left\{(b_2, b_3) \in \mathbb{R}^2 \setminus b_3 = -M, \Gamma_-^g < b_2 < \Gamma_+^g\right\} \\
\Gamma^s & := & \left\{(b_2, b_3) \in \mathbb{R}^2 \setminus \exists \underline{\zeta} \in \mathcal{H}, \Delta(\underline{\zeta}) = \partial_\sigma \Delta(\underline{\zeta}) = 0\right\} \\
& = & \left\{(b_2, b_3) \in \mathbb{R}^2 \setminus b_3 > -M, b_2^2 + b_3^2 = 1\right\} \\
SU & := & \left\{(b_2, b_3) \in \mathbb{R}^2 \setminus \exists \underline{\zeta} \in \Xi \setminus \Xi_0, \underline{\zeta} \neq (1, 0), \Delta(\underline{\zeta}) = 0\right\} \\
& = & \left\{(b_2, b_3) \in \mathbb{R}^2 \setminus b_3 > -M, b_2^2 + b_3^2 < 1\right\} \\
SS & := & \left\{(b_2, b_3) \in \mathbb{R}^2 \setminus \forall \zeta \in \Xi, \Delta(\zeta) \neq 0\right\} \\
& = & \left\{(b_2, b_3) \in \mathbb{R}^2 \setminus b_3 < -M, \Gamma_-^g < b_2 < \Gamma_+^g\right\} \\
& \cup & \left\{(b_2, b_3) \in \mathbb{R}^2 \setminus b_3 > -\dfrac{1}{M}, \Gamma_+^g < b_2 < \Gamma_-^g\right\}
\end{array}$$

All other parameters (b_2, b_3) give rise to problems in the so-called WR class, that is for which the Lopatinskii determinant vanishes exactly at first order in the hyperbolic region $\mathcal{H}$.

Remark The transitions described above are those predicted in [2]. The set Γ^w contain frequencies which satisfy the assumption 2.5 and assumption 2.6.

It has already been mentioned in the begining of this paragraph that

$$\mathcal{D} \subset \tilde{\mathcal{G}} \cup SS.$$

But for $(b_2, b_3) \in \tilde{\mathcal{G}}$ the matrix associated to the quadratic form $\langle A_2 X, X\rangle$ is

$$\mathcal{M} = \begin{bmatrix} M\left(1 + \frac{(1+Mb_3)^2}{1-M^2}\right) & \pm\frac{(1+Mb_3)^2}{\sqrt{1-M^2}} \\ \pm\frac{(1+Mb_3)^2}{\sqrt{1-M^2}} & \left(M(1+b_3^2) + 2b_3\right) \end{bmatrix},$$

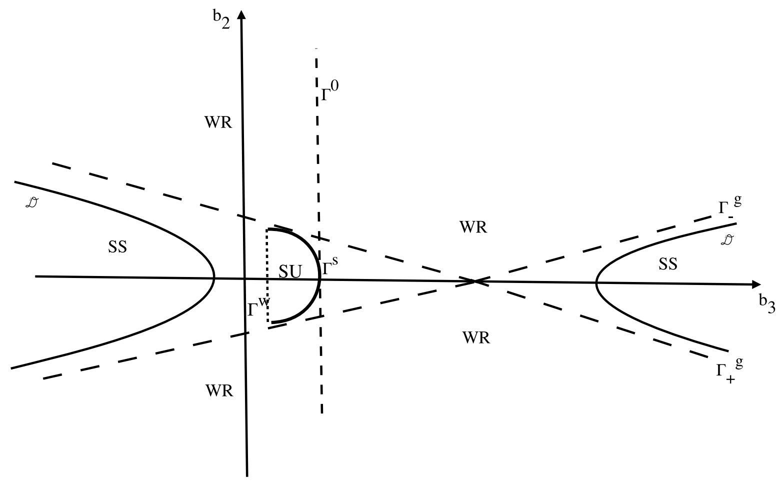
and it is easy to see that $det(\mathcal{M}) < 0$ which means that $\langle A_2 X, X\rangle$ is nonpositive. In other words $\mathcal{D} \cap \tilde{\mathcal{G}}$ is empty.
For $(b_2, b_3) \in SS$ the matrix associated to the quadratic form $\langle A_2 X, X\rangle$ is

$$\mathcal{M}' = \begin{bmatrix} M\left(1 + b_2^2\right) & b_2\left(1 + Mb_3\right) \\ b_2\left(1 + Mb_3\right) & \left(M(1+b_3^2) + 2b_3\right) \end{bmatrix}.$$

Since $M\left(1+b_2^2\right)<0$, $\langle A_2X, X\rangle$ is nonpositive for all X if and only if $det(\mathcal{M}')\geq 0$ that is to say :

$$b_2^2(M^2-1)+(Mb_3+1)^2+(M^2-1)\geq 0.$$

The boundary of $\mathcal{D}$ is a hyperbola, included in SS, for which the corresponding boundary conditions are maximal dissipative though not stricly dissipative. As predicted by theorem 2.5, Γ^w does not meet $\mathcal{D}$.
We summarize the above discussion in the following scheme.



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